Discretization Error Control for Constrained Aerodynamic Shape Optimization

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SIAM CSE 2019
Spokane, Washington, USA
February 25, 2019

Outline

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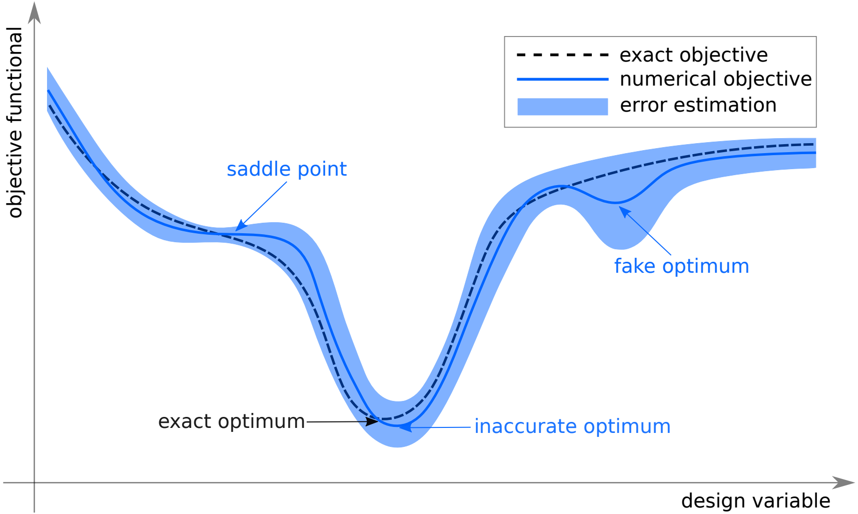
How does discretization error affect optimization?

Design/Optimization: Numerical optimization + CFD analysis



Control the discretization error in the optimization!

Outline

Optimization problem

General optimization problem

- Determine the design variables $\mathbf{x}$ that minimize the objective function J^{adapt}:

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & J^{\text{adapt}}(\mathbf{U}, \mathbf{x}) \\
\text{s.t.} \quad & \mathbf{R}(\mathbf{U}, \mathbf{x}) = \mathbf{0} \\
& \mathbf{R}^{\text{trim}} = \mathbf{J}^{\text{trim}} - \bar{\mathbf{J}}^{\text{trim}} = \mathbf{0}
\end{aligned}
$$

- Objective output J^{adapt}: directly targeted for mesh adaptation
- Trim outputs $\mathbf{J}^{\text{trim}}$: indirectly affect the objective error
- $\bar{\mathbf{J}}^{\text{trim}}$: constant target values for trim outputs
- $\mathbf{R}(\mathbf{U}, \mathbf{x})$: governing PDEs, e.g. flow equations
- Solve $\mathbf{R}(\mathbf{U}, \mathbf{x})$ to get the (flow) states solution $\mathbf{U}$

Optimality condition

Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{U},\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = J^{\text{adapt}}(\mathbf{U},\mathbf{x}) + \boldsymbol{\lambda}^T\mathbf{R}(\mathbf{U},\mathbf{x}) + \boldsymbol{\mu}^T\mathbf{R}^{\text{trim}}(\mathbf{U},\mathbf{x})$$

First-order optimality (Karush-Kuhn-Tucker) condition:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial\mathbf{x}} + \boldsymbol{\lambda}^T\frac{\partial\mathbf{R}}{\partial\mathbf{x}} + \boldsymbol{\mu}^T\frac{\partial\mathbf{R}^{\text{trim}}}{\partial\mathbf{x}} = \mathbf{0} \qquad \text{optimal design}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial\mathbf{U}} + \boldsymbol{\lambda}^T\frac{\partial\mathbf{R}}{\partial\mathbf{U}} + \boldsymbol{\mu}^T\frac{\partial\mathbf{R}^{\text{trim}}}{\partial\mathbf{U}} = \mathbf{0} \qquad \text{coupled adjoint}$$

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{\lambda}} = \mathbf{R}(\mathbf{U},\mathbf{x}) = \mathbf{0} \qquad \text{physical feasibility}$$

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{\mu}} = \mathbf{R}^{\text{trim}}(\mathbf{U},\mathbf{x}) = \mathbf{0} \qquad \text{trim condition}$$

Flow PDE solve, always physically feasible: $\frac{\partial\mathcal{L}}{\partial\boldsymbol{\lambda}} = \mathbf{0}$

Choose coupled adjoint variables $\boldsymbol{\lambda}$, such that, $\frac{\partial\mathcal{L}}{\partial\mathbf{U}} = \mathbf{0}$

Coupled adjoints in constrained optimization

Coupled adjoints in constrained optimization:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0}$$

$$\Rightarrow \boldsymbol{\lambda}^T = -\left(\frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} \right) \frac{\partial \mathbf{R}}{\partial \mathbf{U}}^{-1} = (\boldsymbol{\Psi}^{\text{adapt}} + \boldsymbol{\Psi}^{\text{trim}} \boldsymbol{\mu})^T$$

Sensitivity analysis:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{x}} &= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} \\
&= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + (\boldsymbol{\Psi}^{\text{adapt}} + \boldsymbol{\Psi}^{\text{trim}} \boldsymbol{\mu})^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{J}^{\text{trim}}}{\partial \mathbf{x}} \\
&= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + (\boldsymbol{\Psi}^{\text{adapt}})^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \left[\frac{\partial \mathbf{J}^{\text{trim}}}{\partial \mathbf{x}} + (\boldsymbol{\Psi}^{\text{trim}})^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} \right] \\
&= \frac{dJ^{\text{adapt}}}{d\mathbf{x}} + \boldsymbol{\mu}^T \frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}}
\end{aligned}$$

Reduced optimization problem

Full optimality condition:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} = \mathbf{0} \qquad \text{optimal design}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0} \qquad \text{coupled adjoint}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{R}(\mathbf{U}, \mathbf{x}) = \mathbf{0} \qquad \text{physical feasibility}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mu}} = \mathbf{R}^{\text{trim}}(\mathbf{U}, \mathbf{x}) = \mathbf{0} \qquad \text{trim condition}$$

Reduced optimization problem

Full optimality condition:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} = \mathbf{0}$$ optimal design

$$\frac{\partial \mathcal{L}}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0}$$ ~~coupled adjoint~~

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} = \mathbf{R}(\mathbf{U}, \mathbf{x}) = \mathbf{0}$$ ~~physical feasibility~~

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mu}} = \mathbf{R}^{\text{trim}}(\mathbf{U}, \mathbf{x}) = \mathbf{0}$$ trim condition

Reduced optimality condition (what the optimizer does):

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{dJ^{\text{adapt}}}{d\mathbf{x}} + \boldsymbol{\mu}^T \frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}} = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mu}} = \mathbf{R}^{\text{trim}} = \mathbf{0}$$

Outline

Output error estimation for optimization

- Numerical error affects both objective and constraint outputs.

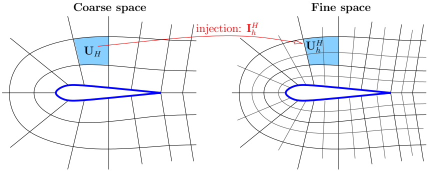
$$\text{coarse space: } \mathbf{x}_0 \rightarrow \text{optimization} \rightarrow \underbrace{\mathbf{U}_H, \mathbf{x}_H^*}_{\text{optimal design}} \rightarrow J_H(\mathbf{U}_H, \mathbf{x}_H^*)$$

$$\text{fine space: } \mathbf{x}_0 \rightarrow \text{optimization} \rightarrow \underbrace{\mathbf{U}_h, \mathbf{x}_h^*}_{\text{optimal design}} \rightarrow J_h(\mathbf{U}_h, \mathbf{x}_h^*)$$

- Objective error estimates for the optimal design,

$$\delta J_{\text{opt}}^{\text{adapt}} = J_H^{\text{adapt}}(\mathbf{U}_H, \mathbf{x}_H^*) - J_h^{\text{adapt}}(\mathbf{U}_h, \mathbf{x}_h^*) = J_h^{\text{adapt}}(\mathbf{U}_h^H, \mathbf{x}_H^*) - J_h^{\text{adapt}}(\mathbf{U}_h, \mathbf{x}_h^*)$$

- State injection: $\mathbf{U}_h^H = \mathbf{I}_h^H \mathbf{U}_H$
- Fine space flow residual,
  $\mathbf{R}_h(\mathbf{U}_h^H, \mathbf{x}_H^*) \neq \mathbf{0}$



- Fine space trim residual,
  $\mathbf{R}_h^{\text{trim}}(\mathbf{U}_h^H, \mathbf{x}_H^*) = \mathbf{J}_h^{\text{trim}}(\mathbf{U}_h^H, \mathbf{x}_H^*) - \bar{\mathbf{J}}^{\text{trim}} = \mathbf{J}_H^{\text{trim}}(\mathbf{U}_H, \mathbf{x}_H^*) - \bar{\mathbf{J}}^{\text{trim}} = \mathbf{0}$

Output error estimation for optimization

Recall the optimality condition:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} = \mathbf{0}$$

Output error estimation for optimization

Recall the optimality condition:

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}}$$

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}}$$

Output error estimation for optimization

Recall the optimality condition:

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}}$$

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}}$$

Error estimation using coupled adjoints

$$\begin{aligned}
\delta J^{\text{adapt}}_{\text{opt}} &= J^{\text{adapt}}_h(\mathbf{U}^H_h, \mathbf{x}^*_H) - J^{\text{adapt}}_h(\mathbf{U}_h, \mathbf{x}^*_h) = \frac{\partial J^{\text{adapt}}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial J^{\text{adapt}}_h}{\partial \mathbf{x}}\delta\mathbf{x} \\
&= -\boldsymbol{\lambda}^T_h \frac{\partial \mathbf{R}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} - \boldsymbol{\mu}^T_h \frac{\partial \mathbf{R}^{\text{trim}}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} - \boldsymbol{\lambda}^T_h \frac{\partial \mathbf{R}_h}{\partial \mathbf{x}}\delta\mathbf{x} - \boldsymbol{\mu}^T_h \frac{\partial \mathbf{R}^{\text{trim}}_h}{\partial \mathbf{x}}\delta\mathbf{x} \\
&= -\boldsymbol{\lambda}^T_h \left(\frac{\partial \mathbf{R}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial \mathbf{R}_h}{\partial \mathbf{x}_h}\delta\mathbf{x} \right) - \boldsymbol{\mu}_h \left(\frac{\partial \mathbf{R}^{\text{trim}}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial \mathbf{R}^{\text{trim}}_h}{\partial \mathbf{x}_h}\delta\mathbf{x} \right) \\
&= -\boldsymbol{\lambda}^T_h \delta\mathbf{R}_h - \boldsymbol{\mu}^T_h \delta\mathbf{R}^{\text{trim}}_h
\end{aligned}$$

Output error estimation for optimization

Recall the optimality condition:

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}}$$

$$\frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} = -\boldsymbol{\lambda}^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} - \boldsymbol{\mu}^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}}$$

Error estimation using coupled adjoints

$$\begin{aligned}
\delta J^{\text{adapt}}_{\text{opt}} &= J_h^{\text{adapt}}(\mathbf{U}_h^H, \mathbf{x}_H^*) - J_h^{\text{adapt}}(\mathbf{U}_h, \mathbf{x}_h^*) = \frac{\partial J_h^{\text{adapt}}}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial J_h^{\text{adapt}}}{\partial \mathbf{x}}\delta\mathbf{x} \\
&= -\boldsymbol{\lambda}_h^T \frac{\partial \mathbf{R}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} - \boldsymbol{\mu}_h^T \frac{\partial \mathbf{R}_h^{\text{trim}}}{\partial \mathbf{U}_h}\delta\mathbf{U} - \boldsymbol{\lambda}_h^T \frac{\partial \mathbf{R}_h}{\partial \mathbf{x}}\delta\mathbf{x} - \boldsymbol{\mu}_h^T \frac{\partial \mathbf{R}_h^{\text{trim}}}{\partial \mathbf{x}}\delta\mathbf{x} \\
&= -\boldsymbol{\lambda}_h^T \left(\frac{\partial \mathbf{R}_h}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial \mathbf{R}_h}{\partial \mathbf{x}_h}\delta\mathbf{x} \right) - \boldsymbol{\mu}_h \left(\frac{\partial \mathbf{R}_h^{\text{trim}}}{\partial \mathbf{U}_h}\delta\mathbf{U} + \frac{\partial \mathbf{R}_h^{\text{trim}}}{\partial \mathbf{x}_h}\delta\mathbf{x} \right) \\
&= -\boldsymbol{\lambda}_h^T \delta\mathbf{R}_h - \boldsymbol{\mu}_h^T \delta\mathbf{R}_h^{\text{trim}} \\
&= -\boldsymbol{\lambda}_h^T \mathbf{R}_h(\mathbf{U}_h^H, \mathbf{x}_H^*)
\end{aligned}$$

Output error estimation for optimization

Objective error estimates for the optimal design,

$$\begin{aligned}\delta J_{\text{opt}}^{\text{adapt}} &= -\boldsymbol{\lambda}_h^T \mathbf{R}_h(\mathbf{U}_h^H, \mathbf{x}_H^*) \\ &= -(\boldsymbol{\Psi}_h^{\text{adapt}})^T \mathbf{R}_h(\mathbf{U}_h^H, \mathbf{x}_H^*) - \boldsymbol{\mu}_h^T \left(\boldsymbol{\Psi}_h^{\text{trim}}\right)^T \mathbf{R}_h(\mathbf{U}_h^H, \mathbf{x}_H^*) \\ &= \underbrace{\delta J^{\text{adapt}}(\mathbf{x}_H^*)}_{\text{objective error only}} + \underbrace{\boldsymbol{\mu}_h^T \delta \mathbf{J}^{\text{trim}}(\mathbf{x}_H^*)}_{\text{inexact constraints satisfaction}}\end{aligned}$$

Error localization:

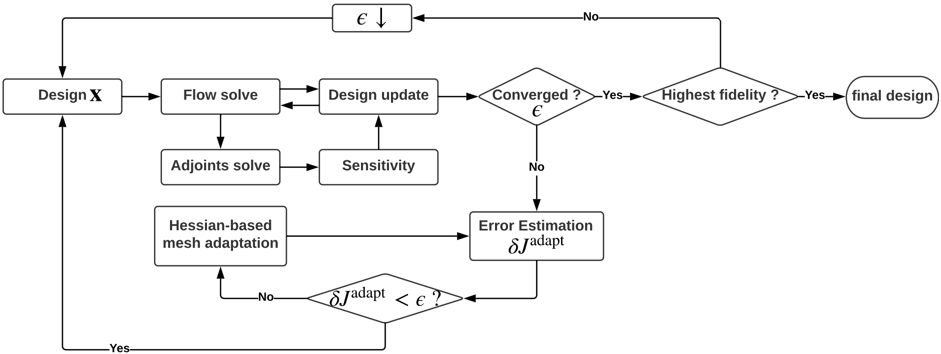
- $\boldsymbol{\Psi}_h$: reconstruction using the coarse-space adjoints $\boldsymbol{\Psi}_H$
- $\boldsymbol{\mu}_h$: extracted from the optimizer on the coarse space
- Adapt (error) indicator: $\eta_e = |\boldsymbol{\Psi}_{h,e}^T \mathbf{R}_{h,e}(\mathbf{U}_h^H, \mathbf{x}_H)|$
- Combined indicator: $\eta_{e,\text{opt}} = \eta_e^{\text{adapt}} + |\boldsymbol{\mu}|^T \boldsymbol{\eta}_e^{\text{trim}}$

Mesh adaptation:

- Hessian-based adaptation: refine mesh to control error
- Mesh optimization: optimize mesh (minimize error) given a fixed cost, cost $= \dim(\mathbf{U})$

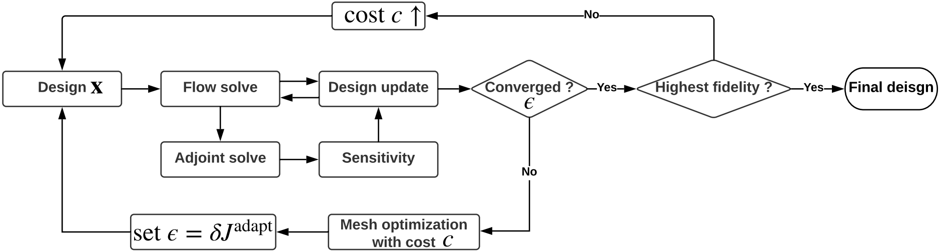
Adaptive multi-fidelity optimization

Error based multi-fidelity optimization:
max allowable objective error $\downarrow \Rightarrow$ mesh refinement $\Rightarrow$ fidelity $\uparrow$



Adaptive multi-fidelity optimization

Cost based multi-fidelity optimization:
given cost $\uparrow \Rightarrow$ mesh optimization $\Rightarrow$ objective error $\downarrow \Rightarrow$ fidelity $\uparrow$



Outline

1D advection diffusion problem

Governing equation

$$a\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in [0, L],\ u(0) = 0,\ u(L) = 1$$

where Peclet number can be defined as $Pe \equiv aL/\nu$
Analytic solution is given by

$$u(x) = \frac{\exp(Pe \times x/L) - 1}{\exp(Pe) - 1} \qquad \frac{\partial u}{\partial x} = \frac{Pe}{L}\frac{\exp(Pe \times x/L)}{\exp(Pe) - 1}$$

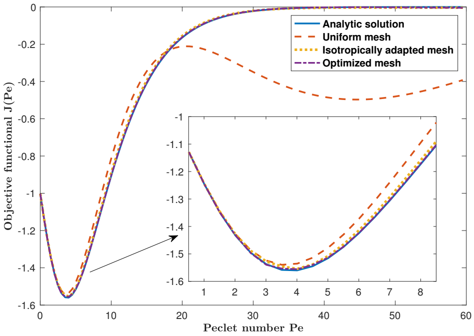
Optimization problem

$$\min_{\mathbf{x}=Pe} \quad J(\mathbf{x}) = -\left.\frac{\partial u}{\partial x}\right|_{x=0.76L}$$

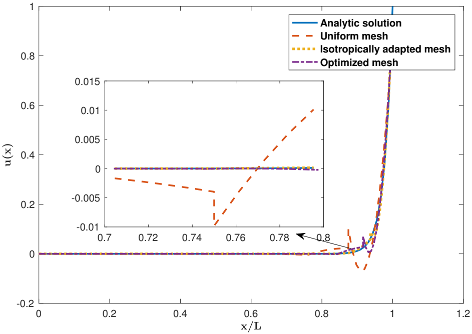
$$\text{s.t.} \quad a\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$$

Analytic solution: $\mathbf{x}^* = 3.8060, \quad \mathcal{J}(\mathbf{x}^*) = -1.5615$

Optimization on different meshes ($DG, p = 2, N_e = 8$)



objective on different meshes



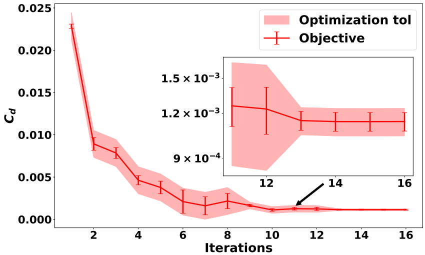
states solution u at spurious optima

Initial design	Mesh	$\|\mathbf{x}_H^* - \mathbf{x}^*\|$	$\|\delta J_H(\mathbf{x}_H^*)\|$	$\|J_H(\mathbf{x}_H^*) - \mathcal{J}(\mathbf{x}^*)\|$
$\mathbf{x}_0 = 40$	uniform	40.79686	0.15555	1.06446
	iso-adapted	0.08455	0.01309	0.01423
	optimized	0.00400	0.00063	0.00072
$\mathbf{x}_0 = 20$	uniform	0.16022	0.02063	0.02109
	iso-adapted	0.03803	0.00505	0.00525
	optimized	0.00400	0.00063	0.00072

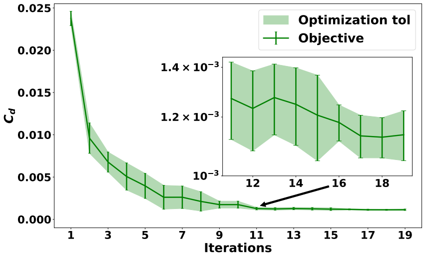
Inviscid transonic airfoil optimization

NACA 0012, $M_\infty = 0.8$, $\alpha_0 = 1.25°$

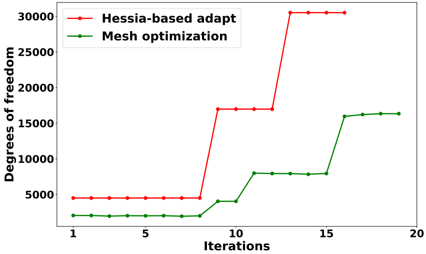
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $\bar{J}^{\text{trim}} = 0.4$, $A \geq A_{\min}$



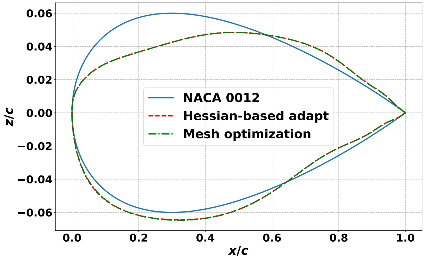
objective convergence (Hessian-based)



objective convergence (Mesh optimization)

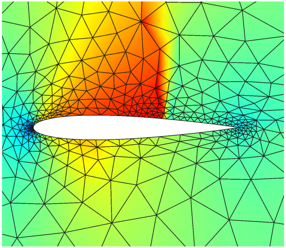


mesh size evolution

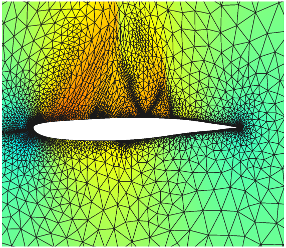


initial and final designs

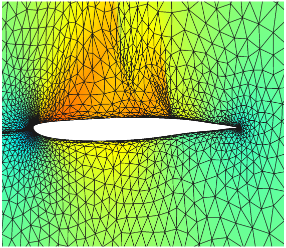
Inviscid transonic airfoil optimization



initial design



optimized design
(Hessian-based adaptation)

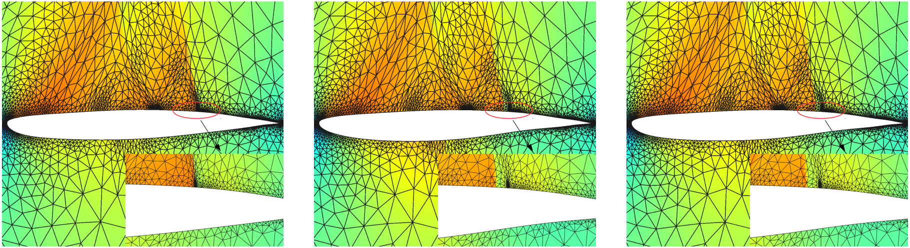


optimized design
(mesh optimization)

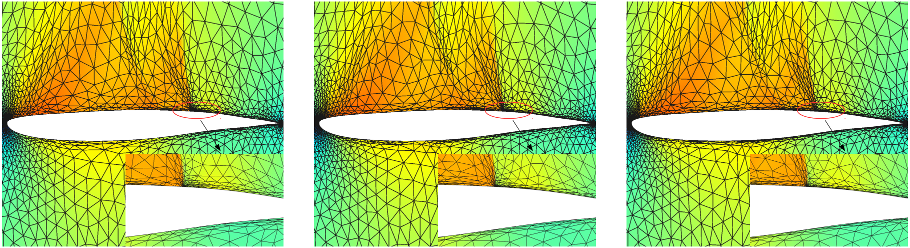
Optimization on different meshes ($DG, p = 1$, opt tol $= 1 \times 10^{-4}$)

	$c_{d,0}$	$c_{d,\mathrm{opt}}$	$\delta c_{d,\mathrm{opt}}$	cost (dof)
Hessian-based	2.242E−2	1.140E−3	6.708E−5	$\sim$ 30000
Mesh optimization		1.136E−3	8.752E−5	$\sim$ 15000

Why mesh optimization in shape optimization ?



Hessian-based adapted meshes (9^{th}, 10^{th}, 12^{th} optimization step from left to right)



optimized meshes (16^{th}, 17^{th}, 19^{th} optimization step from left to right)

Outline

Conclusions and future work

Conclusions:

- Numerical error should be carefully controlled
- Traditional aerodynamic optimization
  a priori mesh, numerical error not controlled
- Error estimation + mesh adaptation
  actively controls the numerical error
- Prevent over-refining and over-optimizing during optimization

Future Work:

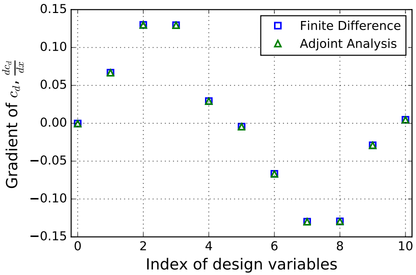
- Incorporate with sensitivity error estimates
- Accelerate error estimation and mesh adaptation process
- Combine with h-p refinement

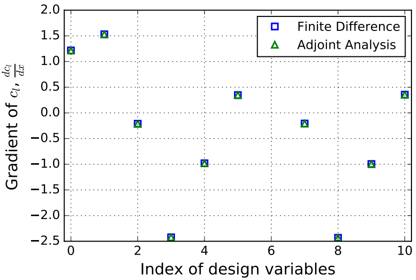
Acknowledgments

Department of Energy
DE-FG02-13ER26146/DE-SC0010341
Boeing Company, with technical monitor Dr. Mori Mani

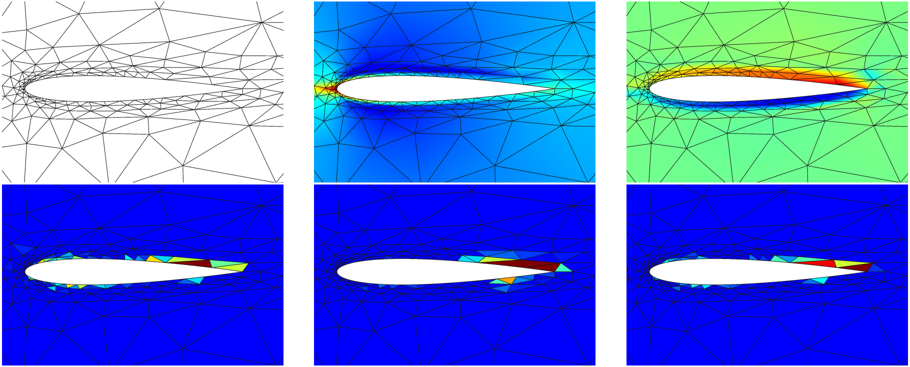
— Thank you —

Sensitivity verification



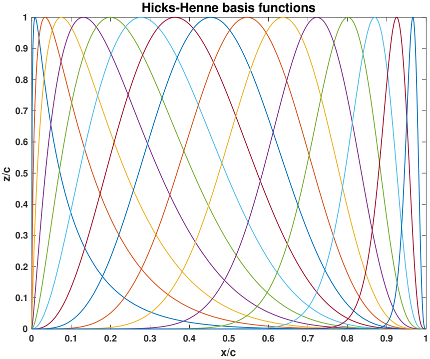


Adapt indicator



Airfoil parameterization

- Hicks-Henne basis functions: linear combination of "bump" functions added to the baseline airfoil



$$z = z_{\text{base}} + \sum_{i=0}^{n} a_i \phi_i(x)$$

$$\phi_i(x) = \sin^{t_i}(\pi x^{m_i})$$

$$m_i = \ln(0.5)/\ln(x_{M_i})$$

x : coord along the airfoil chord
z : vertical surface coord
x_{M_i} : maxima location
t_i : width of the bump function

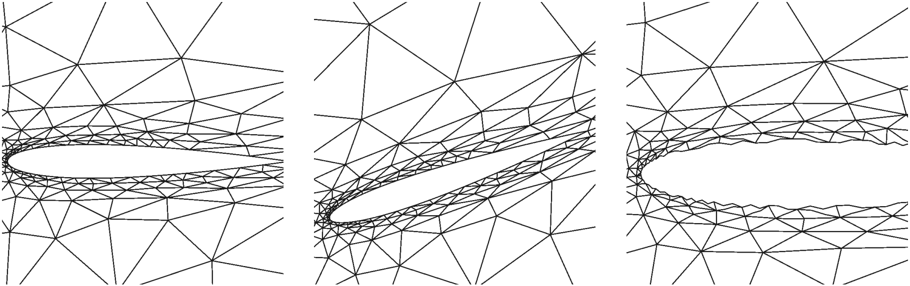
- Design parameters: $\mathbf{x} = [\alpha, a_1, a_2, \cdots, a_n]^T$
  Coefficients of Hicks-Henne basis $+$ angle of attack α

Mesh Movement

- Radial Basis Function (RBF): only depends on the distance from the origin or a center, e.g. $\phi(x) = e^{-x^2}$
- We can use a sum of RBFs $\phi(\|\vec{x}\|)$ and a polynomial $p(\vec{x})$ to interpolate the original function (mesh movement):

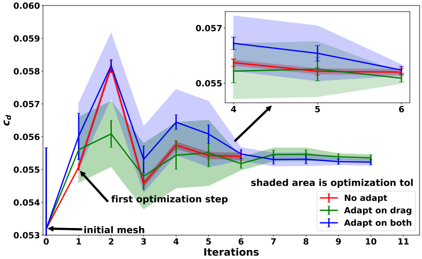
$$\vec{d}(\vec{x}) \approx \tilde{d}(\vec{x}) = \sum_{i=1}^{N_b} \vec{r}_i \phi(\|\vec{x} - \vec{x}_i\|) + \vec{p}(\vec{x})$$

- Solving for a linear system $\mathcal{O}(N_b)$ of $\vec{r}_i$
- Mesh connectivity information not required

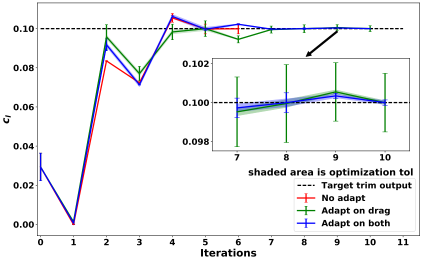


Laminar airfoil optimization

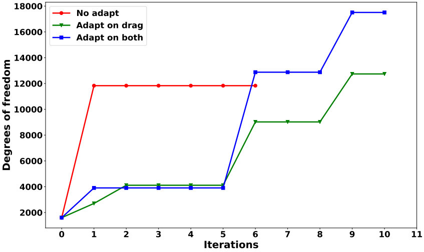
NACA 0012, $Re = 5000$, $M_\infty = 0.5$, $\alpha_0 = 0^\circ$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $\bar{J}^{\text{trim}} = 0.1$, $A \geq A_{min}$



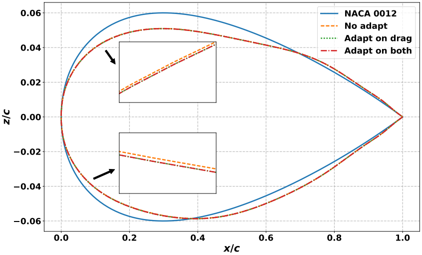
objective convergence histroy



constraint convergence history

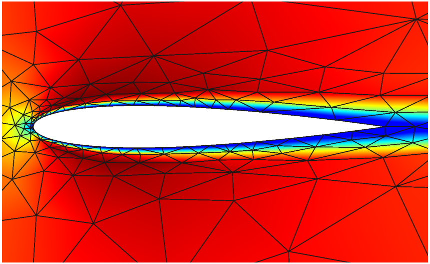


initial and final designs

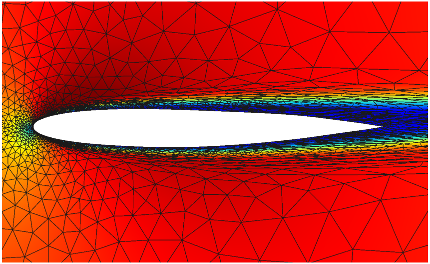


mesh size evolution

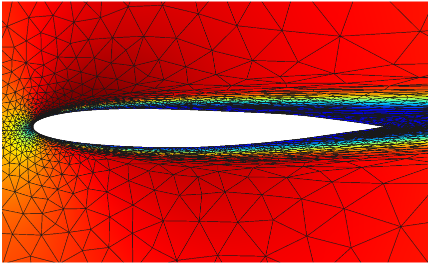
Laminar low-speed airfoil optimization



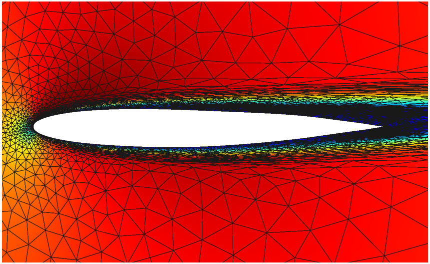
initial design



optimized design (no adapt)



optimized design (adapt on drag)



optimized design (adapt on both)

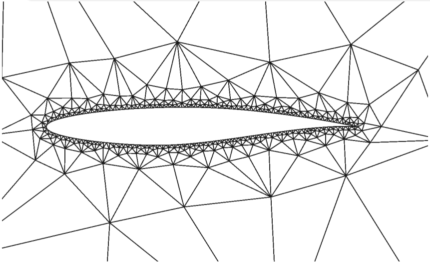
Laminar low-speed airfoil optimization

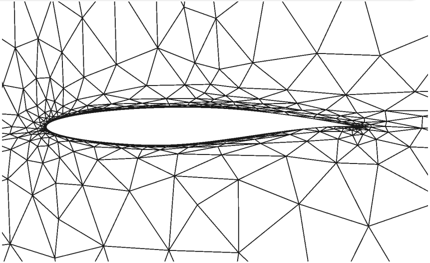
Optimization on different meshes ($DG, p = 1$, opt tol $= 1\text{E}{-4}$)

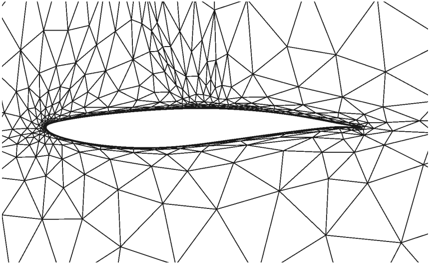
	δJ^{adapt}	$\boldsymbol{\mu}_h^T \delta J^{\text{trim}}$	$\delta J^{\text{adapt}}_{\text{opt}}$	$\left\lVert \left(J^{\text{adapt}}_{h,p}\right)^* - \left(J^{\text{adapt}}_{h,p+1}\right)^* \right\rVert$
Fixed mesh	$1.347\text{E}{-4}$	$1.456\text{E}{-4}$	$2.803\text{E}{-4}$	$2.013\text{E}{-4}$
Adapt on drag	$9.060\text{E}{-5}$	$8.490\text{E}{-5}$	$1.755\text{E}{-4}$	$1.758\text{E}{-4}$
Adapt on both	$8.402\text{E}{-5}$	$7.009\text{E}{-6}$	$9.103\text{E}{-5}$	$9.160\text{E}{-5}$

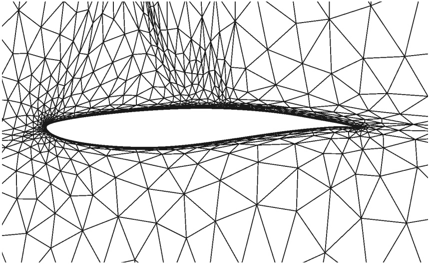
Turbulent transonic airfoil optimization

RAE 2822, $Re = 6.5 \times 10^6$, $M_\infty = 0.734$, $\alpha_0 = 2.79°$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $\bar{J}^{\text{trim}} = 0.824$, $A \geq A_{min}$

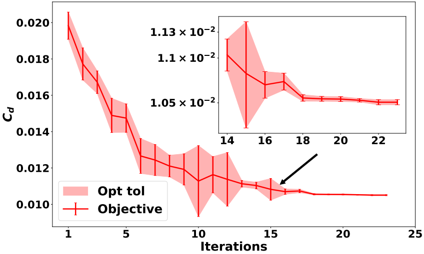

initial mesh


mesh at the first optimization step

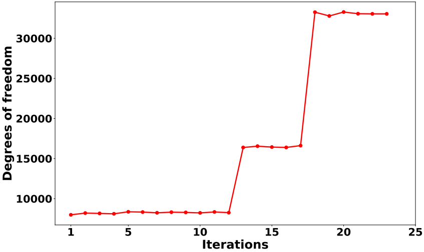

mesh at one intermediate optimization step


final mesh

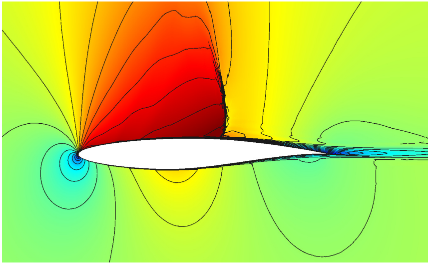
Turbulent transonic airfoil optimization



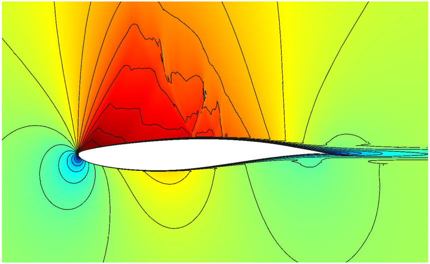
objective convergence



mesh size evolution



initial design

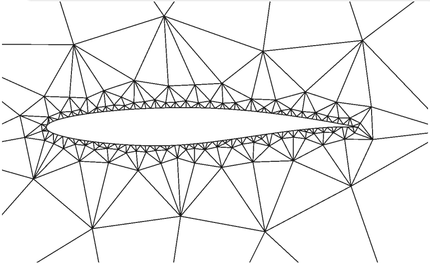


optimized design

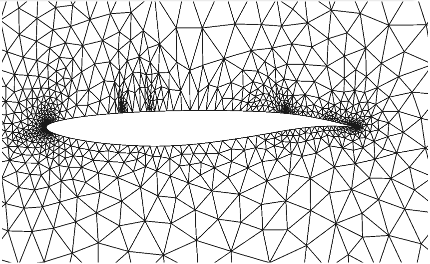
Multi-point optimization

RAE 2822, $M_{1,\infty} = 0.72$, $M_{2,\infty} = 0.76$, $\alpha_0 = 2.79^\circ$

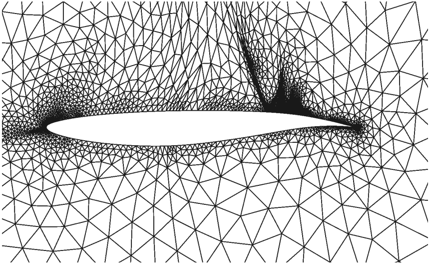
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $\bar{J}_1^{\text{trim}} = \bar{J}_2^{\text{trim}} = 0.75$, $A \geq A_{\text{min}}$



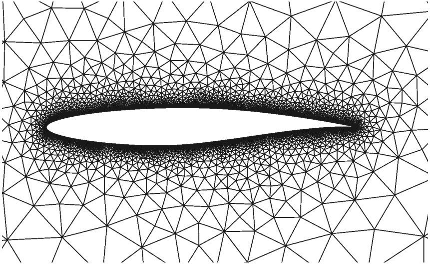
initial mesh for multifidelity optimization



final mesh at $M = 0.72$

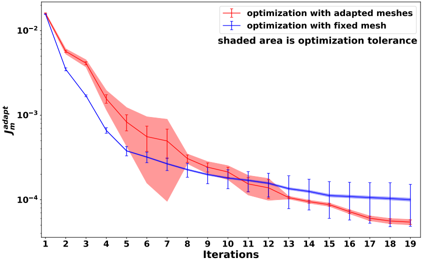


final mesh at $M = 0.76$

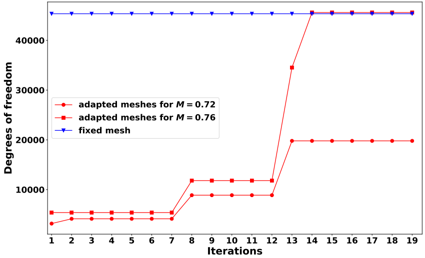


mesh for fixed fidelity optimization

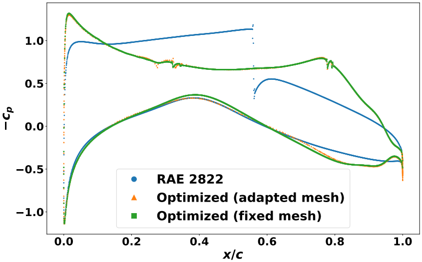
Multi-point optimization



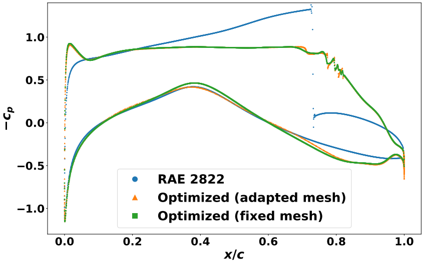
objective convergence



mesh evolution



pressure distribution at $M = 0.72$



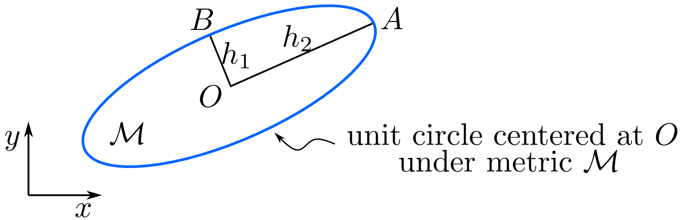
pressure distribution at $M = 0.76$

Unstructured mesh adaptation and mesh optimization

Riemannian metric field $\mathcal{M} \in \mathbb{R}^{d\times d}$

Encode the discrete mesh (shape and size) with a continuous symmetric positive definite (SPD) tensor field, $\mathcal{M}(\vec{x}) \in \mathbb{R}^{d\times d}$

distance under the metric from $\vec{x}$ to $\vec{x} + \delta\vec{x}$, $\delta l_{\mathcal{M}} = \sqrt{\delta\vec{x}^T \mathcal{M} \delta\vec{x}}$



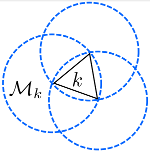
- Eigenvectors $\Rightarrow$ principle stretching direction
- Eigenvalues $\lambda_i = 1/h_i^2 \Rightarrow$ stretching magnitude

Unstructured mesh adaptation and mesh optimization

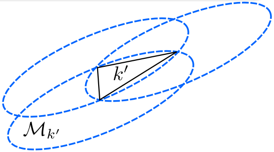
Metric conforming mesh

A mesh in which each edge has the same length under the metric

Isotropic metric:



Anisotropic metric:



Mesh adaptation and mesh optimization techniques

- Hessian-based adaptation: refine mesh to control error
  solution Hessian $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $\Rightarrow$ eigenvectors $\Rightarrow$ element shape
  error estimates $\eta_{e,\text{opt}}$ $\Rightarrow$ eigenvalues magnitue $\Rightarrow$ mesh size
- Mesh optimization: optimize mesh given a fixed cost
  Cost: $\dim(\mathbf{U})$ $\Rightarrow$ degrees of freedom (dof)

$$\min_{\mathcal{M}(\vec{x})} \sum \eta_{e,\text{opt}} \qquad \text{s.t.} \quad \text{cost} = \text{const}$$